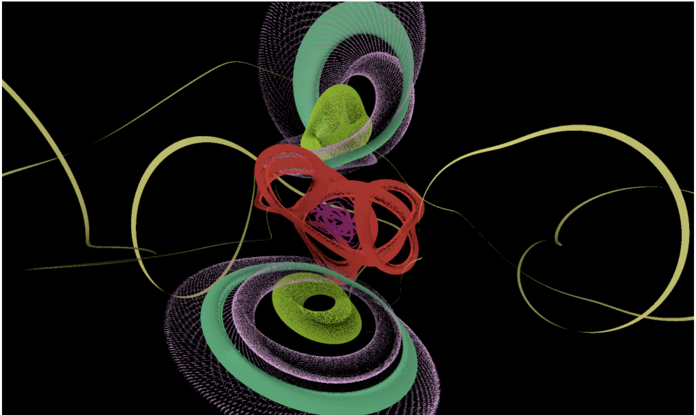
Introduction to the Dynamics of Rational Surface Automorphisms

Eric Bedford

Stony Brook University



Notation and terminology: Projective space, rational maps

projective space: $\mathbb{P}^2 = \{[x_0 : x_1 : x_2] = [\lambda x_0 : \lambda x_1 : \lambda x_2], \lambda \neq 0\}$

rational map $f = [f_0 : f_1 : f_2] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, f_j polynomials of same degree d, no common factor. We define the *degree* of the map as:

$\deg(f) := d = \deg(f_0) = \deg(f_1) = \deg(f_2)$

Iteration is "normal", except that we need to cancel common factors each time. In fact, when we do this, the degree can drop by a lot – even to 1. Thus determining degree growth is not obvious.

We may view $\mathbb{C}^2 \subset \mathbb{P}^2$ via the map $(x, y) \mapsto [1 : x : y]$. Any rational map may also be given by $R = \left(\frac{P_1}{Q_1}, \frac{P_2}{Q_2}\right) : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$

f is *birational* if it has an inverse f^{-1}, $f \circ f^{-1} = \mathrm{id}$ where defined.

Dynamical degree

Behavior under composition: $\deg(f \circ g) \leq \deg(f)\deg(g)$

Equality may fail: Cremona Involution

$$\sigma(x, y) = \left(\frac{1}{x}, \frac{1}{y}\right) : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$$

$$\deg(\sigma) = 2, \quad \deg(\sigma^2) = 1$$

As a map of $\mathbb{P}^2$, this is written $\sigma([x_0 : x_1 : x_2]) = [x_1 x_2, x_0 x_2, x_0 x_1]$,

$$\sigma^2 = \mathrm{id} = x_0 x_1 x_2 [x_0 : x_1 : x_2]$$

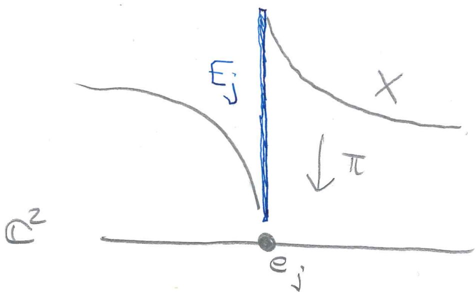
$\deg(f)$ is not invariant under birational conjugacy, so we define

$$\mathrm{ddeg}(f) = \lim_{n \to \infty} (\deg(f^n))^{1/n}$$

Define *Rational Surface Automorphism* by Theorem

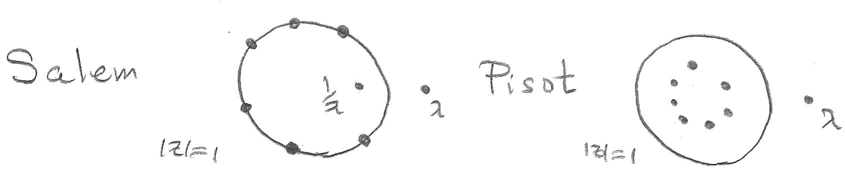
Theorem (Nagata)

Let $F : X \to X$ be a rational surface automorphism such that the action of F^* on $H^2(X)$ has infinite order. Then there is an (iterated) blowup $\pi : X \to \mathbb{P}^2$ and a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\pi \circ F = f \circ \pi$ (or $F = \pi^{-1} \circ f \circ \pi$).



Dynamical degrees: Salem or Pisot

Let $\lambda > 1$ be an algebraic number.
λ is *Salem* if its Galois conjugates are $1/\lambda$, or modulus 1.
λ is a *Pisot* if its Galois conjugates all have modulus < 1.



Theorem (Diller-Favre, Blanc-Cantat)

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map with $ddeg(f) > 1$. Then f induces a rational surface map if and only if $ddeg(f)$ is a Salem number. Otherwise, it is a Pisot number. In either case, it is algebraic. If f is birationally conjugate to an automorphism, then $ddeg(f)$ is irrational.

In later discussion, the automorphisms will be "parametrized" by the Galois conjugates of Salem numbers.

Why consider only rational surface X?

Theorem (Cantat)

Let F be an automorphism of a compact, complex surface X. If F^* has infinite order, then (after possible blow-downs) either:

- $X = \mathbb{T}^2$ *is a complex torus.*
- X *is* $K3$ *(or certain quotients).*
- X *is rational.*

The examples we discuss in this talk will all be defined over number fields. Rational surfaces come in arbitrarily large families, and not necessarily defined over number fields.

Theorem (B-Kim)

For any k, there are k-parameter families (not isotrivial)
$F_\alpha = F_{\alpha_1,\dots,\alpha_k} \in Aut(X_\alpha)$, $\alpha_j \in \mathbb{C}$ *with* $ddeg(F_\alpha) > 1$.

Summary of general dynamical properties: 1

We let $F : X \to X$ be a rational surface automorphism with $\lambda := \mathrm{ddeg}(F) > 1$. Let β be a Kähler form on X with $\int \beta \wedge \beta = 1$.

- There exists a unique $\Theta^{\pm} \in H^2(X; \mathbb{C})$ such that $\Theta^{\pm} \cdot \beta = 1$ and

$$F^*(\Theta^{\pm}) = \lambda^{\pm 1}\Theta^{\pm}$$

  Comment: $\Theta^{\pm} \cdot \Theta^{\pm} = 0$

- There is a unique positive, closed current $T^{\pm}$ in the class $\Theta^{\pm}$. Further, this current is obtained as

$$\frac{1}{\lambda^n}(F^*)^{\pm n}\beta \to T^{\pm}$$

  There is a continuous function $g^{\pm}$ such that $T^{\pm} = \Theta^{\pm} + dd^c g^{\pm}$.

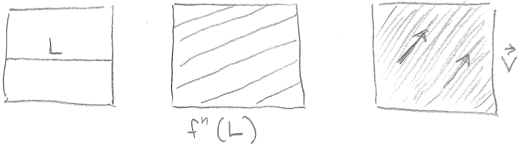
Example of the torus (not rational surface)

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$$

L is an oriented horizontal line. λ is the expanding eigenvalue of A, and $\vec{v}$ is the corresponding eigenvector.

Entropy $\log(\lambda)$ corresponds to the length growth: $\text{Length}(A^n(L)) \sim \lambda^n$

The normalized current $\lambda^{-n}[A^n(L)]$ converges to a constant times $\vec{v}\, dArea$. (k-dimensional currents are represented by a k-vector times a distribution.)



Summary of general dynamical properties: 2

- $\mu := T^+ \wedge T^-$ is the unique measure of maximal entropy $= \log \lambda$.
- μ gives the asymptotic distribution of saddle (periodic) points.
- If p is a saddle (periodic) point, and if $W^{s/u}(p)$ is not algebraic, then $p \in \mathrm{supp}(\mu)$.
- $|\text{Lyapunov exponents}| \geq \frac{\log(\lambda)}{2} > 0$
- $T^\pm$ has laminar structure given by $\mathcal{W}^{s/u}$
- Julia sets $J^\pm = \mathrm{supp}(T^\pm)$ (modulo a finite number of invariant algebraic curves)

Problem

Is it ever the case that supp(μ) is a hyperbolic set? Or is not?

Problem

Is there some rational surface map $F : X \to X$ for which we can describe $J^\pm$ or J explicitly? For instance, can we get a horseshoe-like map?

Simpliest maps: $f_{a,b}(x,y) = \left(y, \frac{y+a}{x+b}\right)$

Theorem (B-Kim, McMullen)

1. *$f_{a,b}$ gives a surface automorphism $\Leftrightarrow$ there exists $n \geq 0$ such that*

$$f^n_{a,b}(-a,0) = (-b,-a)$$

2. *This equation has solutions for every n.*
3. *$ddeg(f_{a,b}) > 1 \leftrightarrow n \geq 7$.*

Theorem (B-Kim)

There exist automorphisms $f_{a,b}$ without invariant curve.

However, the assumption of an invariant curve will make it easier to discuss the existence of automorphisms.

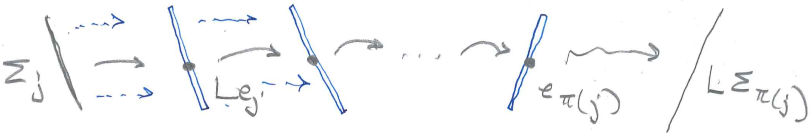
General birational maps of degree 2

Theorem

A birational map of degree 2 is linearly conjugate to $L \circ \iota$, where $L \in PGL(3, \mathbb{C})$, and ι denotes one of the 3 involutions: σ, ρ, τ.

Let $f = L \circ \sigma$ be a rational surface automorphism. We define *orbit data* $((n_0, n_1, n_2), \pi)$, where π is a permutation of $\{0, 1, 2\}$, and

$$\Sigma_j := \{x_j = 0\} \mapsto p_{j,1} := L(e_j) \mapsto p_{j,2} \mapsto \cdots \mapsto p_{j,n_j-1} \mapsto e_{\pi(j)} = p_{j,n_j}$$



Facts: (1) $f_{a,b}$ is conjugate to $L \circ \sigma$ and has orbit data ((1,1,8), cyclic)
(2) If f has positive entropy, then $n_1 + n_2 + n_3 \geq 10$.

Automorphisms with invariant curves: Diller approach

Let $\varphi : \mathbb{C} \to \mathcal{C},\ \varphi(\zeta) = (\zeta, \zeta^3)$ be the cubic with a cusp at infinity.

We start with birational maps of degree 2.

Theorem

For each $\lambda \in \mathbb{C} - \{0\}$, there are 3×3 matrices $S = S_\lambda$, $T = T_\lambda$ such that $f_\lambda := S \circ \sigma \circ T^{-1}$ preserves $\mathcal{C}$, and

$$f_\lambda|_{\mathcal{C}} : \zeta \mapsto \lambda(\zeta - 1) + 1$$

The strategy is to find λ such that the birational map f_λ is actually a rational surface automorphism. If this is the case, then all blowup points will be in $\mathcal{C}$. Thus any point of blowup will be of the form $\varphi(\zeta)$, and it becomes the problem of solving for $\zeta \in \mathbb{C}$.

Quadratic maps with invariant curves: Existence

Theorem (Diller)

Let orbit data $((n_0, n_1, n_2), \pi)$ be given. Except for some specific cases, there is an automorphism $f = L_1 \circ \sigma \circ L_2$ preserving $\mathcal{C}$ which realizes these data. Further, $f|_{\mathcal{C}} : \zeta \mapsto \delta(\zeta - 1) + 1$, where δ is any Galois conjugate to ddeg(f).

Refering to the previous picture, the points of indeterminacy are $T(e_j) = \varphi(\zeta_j^-)$, and the critical image points are $S(e_j) = p_{j,1} = \varphi(\zeta_j^+)$. Using the fact that $\varphi^n(\zeta) = \delta^n(\zeta - 1) + 1$, we are able to solve algebraically for the values of δ and $\zeta_j^\pm$, $j = 0, 1, 2$.

This produces a Salem polynomial for δ; the ζ_j^+ and ζ_j^- are rational functions of δ.

Theorem (Summary)

The Galois conjugates of $d = ddeg(f)$ are:
$d, d^{-1},$ and $|\delta_1| = \cdots = |\delta_{n'}| = 1$, and each of these Galois conjugates gives an automorphism.

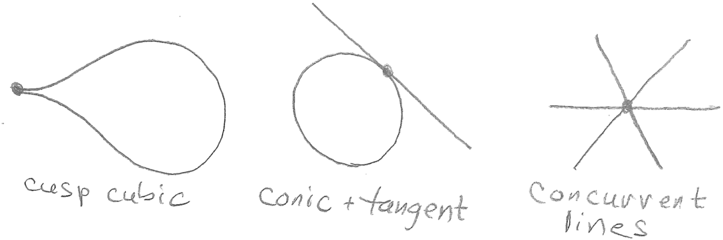
Quadratic maps with invariant curves: Fatou set

The forward/backward *Fatou set* $\mathcal{F}^{\pm}$ is where the iterates of $f^{\pm 1}$ are locally equicontinuous.

Theorem

If there is an invariant curve, then the Fatou set is nonempty.

The possibilities for invariant curves (Diller paper covers all three):



For instance: In the case of the cusp cubic, the cusp point is fixed and has multipliers δ^{-2}, δ^{-3}. Since δ is Galois conjugate of a Salem number, the map is linearizable in a neighborhood of the cusp point. Thus it belongs to $\mathcal{F}^{+}$ or $\mathcal{F}^{-}$ (or both, if $|\delta| = 1$).

Is the other fixed point linearizable?

There are two fixed points on the cusp cubic $\mathcal{C}$. The multipliers at the other fixed point are δ and $\delta^{3-n_1-n_2-n_3}$, so there is a resonance here.

In computer pictures, it seems that the Fatou set contains $\mathcal{C}$, which suggests that the fixed point is linearizable.

Problem

Can f be linearized at this fixed point?

If so, then there is a rank 1 Fatou component Ω containing the whole curve $\mathcal{C}$. It follows that the (meromorphic) volume form is bounded on $X - \Omega$.

If so, we conclude that $X - \Omega$ has finite volume, so in the conservative case, *all Fatou components are periodic*.

Invariant volume form (with poles)

If $\mathcal{C} = \{p(x,y) = 0\}$ is an invariant curve, then

$$\eta := \frac{dx \wedge dy}{p(x,y)} \text{ is invariant: } f^*\eta = c\,\eta \text{ for some } c \in \mathbb{C}$$

For the maps f_δ constructed in the previous Theorem, we have $c = \delta$. These maps are either conservative or dissipative.

Theorem (McMullen, B-Kim)

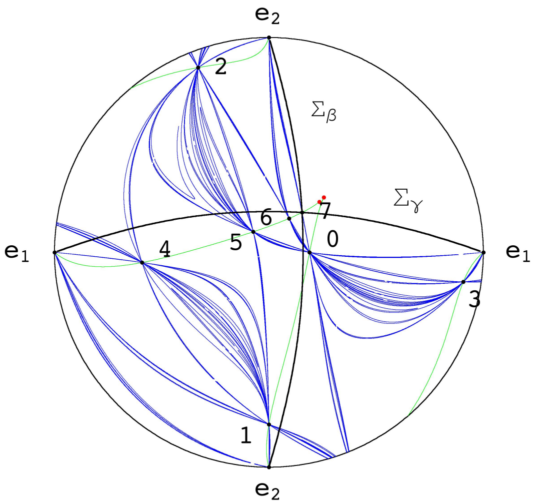
Suppose that $\mathcal{C}$ is invariant and $\delta = ddeg(f) > 1$. Then the cusp at infinity is an attracting fixed point and its basin $\mathcal{B}$ has full volume in the sense that $Vol_\eta(X - \mathcal{B}) = 0$. Since δ is real, f induces a diffeomorphism of the real points X_R. The cusp point has a real basin $\mathcal{B}_R$ inside X_R, and $X_R - \mathcal{B}_R$ has zero area.

Problem

Describe the attractors $\mathcal{A} := X - \mathcal{B}$ and $\mathcal{A}_R := X_R - \mathcal{A}_R$.

Attempt to draw the current of an attractor in $\mathbb{RP}^2$

Invariant cubic in green. Repeller is the cusp (red), other fixed point on cubic (red). Blue is forward iterate of a line.



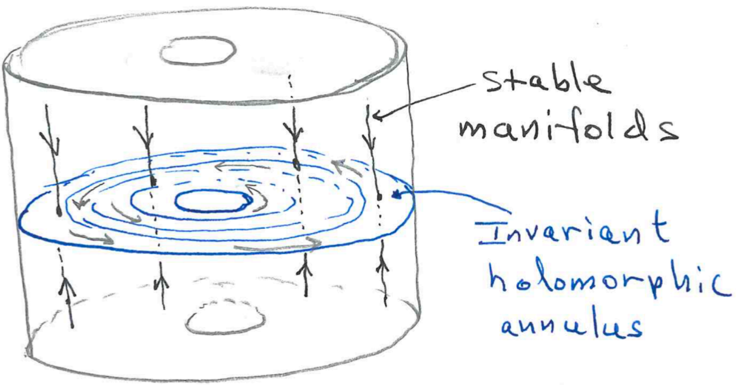
$$\Sigma_\beta \to e_2 \to \Sigma_0 = \text{line at infinity} \to e_1 \to \Sigma_\gamma \to \mathbf{0} \to \mathbf{1} \to \cdots \to \mathbf{7} \to \overline{e_2\mathbf{0}}$$

Model: Attracting Herman ring

For dissipative maps, a rotational annulus or disk will be normally attracting.

$\Omega = A \times \mathbb{C}$; irrational rotation in the annulus $\times$ contraction in $\mathbb{C}$

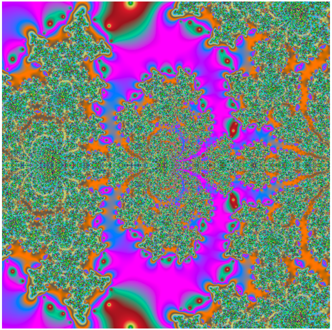
Can such a Fatou component occur for either a complex Hénon map or a rational surface automorphism?

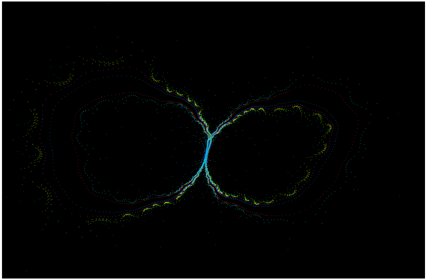


Ushiki: Computer "Example"

Orbit data $((3,3,4),\pi)$, $\pi =$ cyclic also ((2,3,5), cyclic)

Demonstrate this with Ushiki's software.





Left: Complex slice of Julia set. Right: Orbits inside "Herman ring"?

Problem

Can the existence of this apparent Herman ring be proved mathematically?

How to “draw” or “compute” the Fatou set?

Digression: Hénon maps

Have continuous functions $G^{\pm} = \lim_{n\to\infty} \frac{1}{d^n} \log^+ ||f^{\pm n}||$ on $\mathbb{C}^2$.
$J^{\pm} = \partial K^{\pm}$, $J = J^+ \cap J^-$, $K = K^+ \cap K^-$, and the forward/backward Fatou sets are $\mathcal{F}^{\pm} = \mathbb{C}^2 - J^{\pm}$.

Theorem (Friedland-Milnor)

For volume-decreasing (dissipative) Hénon maps, $J^- = \partial K^- = K^-$.
For volume-preserving (conservative) Hénon maps,
$int(K^+) = int(K^-) = int(K)$.

In the hyperbolic, dissipative case, we have $\text{int}(K^+) = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$, union of basins of attraction. Thus in the hyperbolic, dissipative case, the sets $\mathcal{F}^{\pm}$ are “computable”.

Problem

Is the Fatou set “computable” in other cases?
Is the statement “Fatou set $\neq \emptyset$” “computable” for a conservative map?

Rational surface automorphisms

Theorem (Dinh-Sibony, Moncet, Ueda)

$X -$ support$(T^+ + T^-) = \mathcal{F}^+ \cap \mathcal{F}^-$ (modulo an invariant algebraic curve).

In this case, we have no $G^\pm$, so we work with the Lyapunov exponent

$$\Lambda^\pm(p) := \limsup_{n\to\infty} \frac{1}{n} \log ||Df^{\pm n}(p)||$$

Clearly, $\Lambda^\pm = 0$ on $\mathcal{F}^\pm$.

Theorem (Dujardin)

*If $\mu = T^+ \wedge T^-$, and the dynamical degree $\lambda > 1$,
then $\Lambda^\pm(p) \geq \frac{\log(\lambda)}{2}$ for μ a.e. p.*

Theorem

*$\mathcal{F}^+ \cap \mathcal{F}^- =$ interior$(\{\Lambda^+ + \Lambda^- < \frac{\log(\lambda)}{2}\})$
(modulo an invariant algebraic curve).*

Conservative (Volume preserving) maps

Let $\Omega \subset \mathcal{F}^+ \cap \mathcal{F}^-$ be invariant fixed (periodic) component.

$$\mathcal{G} = \{\text{normal limits of subsequences } f^{n_j} \to g : \Omega \to \Omega\}$$

Theorem (B-Kim)

$\mathcal{G}_0$ (connected component of identity in $\mathcal{G}$) $\cong \mathbb{T}^\rho$, $\rho = 1$ or 2.

The Fatou component Ω is a *rotation domain of rank* ρ. It seems that rank 2 is the "generic" case. The Fatou component arising from multipliers δ^{-2}, δ^{-3} at the cusp point, which was noted earlier, has rank $= 1$.

Problem

What sorts of rotation domains Ω can exist? For instance, in the Hénon case, the action on a rank 2 rotation domain is conjugate to a rotation on a Reinhardt domain. Is there a similar model (e.g. canonical toric manifold) for the maps f_δ?

Existence of rotation domains: Fixed points

Theorem (C.L. Siegel)

f may be linearized at a fixed (periodic) point p_0 such that the multipliers of $Df(p_0)$ are sufficiently Diophantine.

Theorem (McMullen, B-Kim)

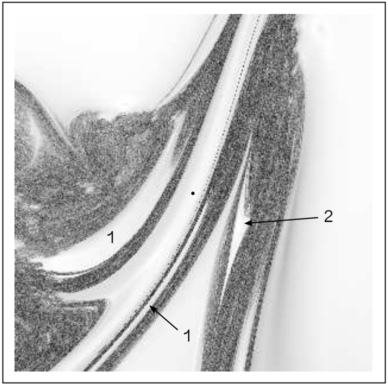
For every dynamical degree in the $f_{a,b} = (y, (y+a)/(x+b))$ family, there is an automorphism with a rank 2 rotation domain, because of fixed (periodic) points with suitable multipliers.

Problem

Is it possible for rotation domains to arise for some reason other than linearization at a fixed point?
Can there exist rotation domains without fixed points?

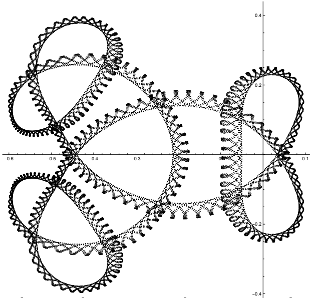
Ushiki example: Another analogue of a Herman ring?

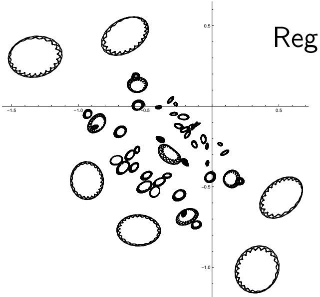
We choose f_δ for a map $f_{a,b}(x,y) = \left(y, \frac{y+a}{x+b}\right)$ with $|\delta| = 1$. Orbit data: ((1,1,8),cyclic).



Complex slice of the Julia set (black) and the Fatou set (white). Detail on right. We will see orbits of points from regions 1 and 2.

Region 1





Region 2.

Looking at the orbits, we have evidence that regions 1 and 2 are in fact in the Fatou set. If this is the case, then these regions are rotation domains with rank either 1 or 2. The closure of a generic point of an invariant Fatou component will be a (real) torus of dimension ρ. The pictures suggests that region 1 is invariant and has rank 2.

The fixed points of f_δ consist of the two fixed points on the invariant curve (in a domain of rank 1), as well as two other points, which are saddles. Thus, region 1 cannot contain a fixed point.

Problem

Can this be proved mathematically?

Invitation – and another picture by Ushiki

Study the "1-parameter" family of rational surface automorphisms

$$f_\delta = S \circ \sigma \circ T^{-1}$$

that preserve a cubic $\mathcal{C}$.

This special quadratic family $\{f_\delta\}$ should be more accessible than the general case, but it contains examples that are nontrivial and interesting.